The values of the lift and drag coefficients that are plotted in Fig 4 do not include the contribution of the spike itself The lift that could be obtained from the deflected spike alone was computed theoretically using Newtonian theory for the part of the spike forward of the separation point for all spike deflections The lift coefficient was found to be less than 4% of the lift coefficient which resulted from the changing pressure distribution over the body Since this value was small, it was neglected in this study

Conclusions

These tests were conducted at one Mach number and one Reynolds' number, and the problems of a high enthalpy Mach 12 flow have not been considered Subject to these restrictions, one may conclude the following: 1) the flow-separation spike can, when deflected, cause forces in the direction of the spike deflection of the same order of magnitude as the drag; 2) the maximum L/D and the maximum lift that result from the deflected spike occur at small angles of spike deflection barring any unexpected effects at large spike-deflection angles; and 3) accurate pressure distributions on the surface of the model can only be obtained if the model is aerodynamically aligned in the tunnel since small amounts of misalignment can greatly alter the measured pressure distribution

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Boundary of Underexpanded Axisymmetric Jets Issuing into Still Air

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THE boundaries of an underexpanded axisymmetric jet issuing supersonically into still air can be calculated by the method of characteristics,[1] but since that is a lengthy procedure, empirical formulas[2] and approximate analytical models have been sought In particular, Adamson and Nicholls[3] and also Love[4] calculated the shape of the free jet in the initial region by requiring that the external streamtube undergo sufficient compressive turning to cancel the local pressure gradient due to area expansion They estimated this gradient from the rate of increase of the local radius of the jet, implying that the flow throughout the jet core is quasi-one-dimensional

A more satisfactory and apparently more accurate model explores fully the fact that the flow along the jet boundary is nearly isentropic, at least initially (turning-compression waves are weak) Under this assumption the jet-edge Mach number is determined entirely by the corner expansion to ambient pressure and remains constant along the jet boundary, together with the associated isentropic flow functions such as the stagnation pressure P_t and the local sonic throat area A^*

Following a procedure similar to that of Ref 3, the condition that the static pressure on the free jet boundary be constant is written as

$$d\left(\frac{P}{P_t}\right) = \frac{\partial(P/P_t)}{\partial(A/A^*)}\frac{dA}{A^*} + \frac{\partial(P/P_t)}{\partial\theta}d\theta \tag{1}$$

$$\frac{dA}{A^*} = 2\left(\frac{A_j}{A^*}\right)\left(\frac{r}{r_j}\right)d\left(\frac{r}{r_j}\right) \tag{2}$$

$$\frac{\partial(P/P_t)}{\partial\theta} = \frac{\gamma_j M^2(P/P_t)}{(M^2-1)^{1/2}} \tag{3}$$

Define a function ψ which according to the present model is a constant along the jet boundary (in contrast with the model of Ref 3):

$$\psi = \frac{\partial \ln(P/P_t)}{\partial(A/A^*)}\frac{A_j}{A^*}\frac{(M^2-1)^{1/2}}{\gamma_j M^2}$$

It can be expressed as follows, using isentropic flow relations,

$$\psi = \frac{2}{\gamma_j+1}\frac{(M^2-1)^{1/2}}{M^2(A/A^*)_M}\left(\frac{A}{A^*}\right)_{M_j} \tag{4}$$

In the forementioned, the area ratios $(A/A^*)_M$ could be written out in terms of Mach number (which is indicated by the subscript) or in terms of the pressure ratios, but they are more conveniently evaluated by the use of compressible flow tables

After integration, Eqs (1–4) yield the following relation:

$$\theta - \theta_0 = \psi[1-(r/r_j)^2] \tag{5}$$

θ_0 is the initial expansion angle at the edge of the nozzle, which is known in terms of the Prandtl-Meyer function and the nozzle divergence half-angle at the exit:

$$\theta_0 = \nu_M - \nu_{Mj} + \theta_j \tag{6}$$

From Eq 5, the shape of the jet boundary in proper generalized coordinates is

$$X = \int_{1/\rho_0}^{R} \text{cotg}[\phi(1-R^2)]dR \tag{7}$$

where

$$X = x/r_j\rho_a \qquad R = r/r_j\rho_a \tag{8}$$

with the following two scale-parameters:

$$\rho_a = r_a/r_j = [1+(\theta_0/\psi)]^{1/2} \tag{9}$$

$$\phi = \psi\rho_a^2 = (\psi+\theta_0) \tag{10}$$

Equation (7) is easily integrated numerically Figure 1 shows several of the generalized jet boundaries with ϕ as a parameter (see also Fig 3) The second parameter ρ_a enters through the definition of the real origin of the free jet (edge of nozzle) X_0, relative to the virtual origin $X = 0$, where

$$X_0 = [X \text{ at } R = 1/\rho_a] = \int_0^{1/\rho a} \text{cotg}[\phi(1-R^2)]dR \tag{11}$$

The shape of the jet boundary is specified by the *two* generalized parameters ϕ and ρ_a, which contain the influence of the *four* physical parameters M_j, θ_j, (P/P_j), and γ_j

Figure 2 shows a comparison of the present analysis with a specific calculation by the method of characteristics[1] and the model of Adamson and Nicholls[3] The results are seen to be remarkably good

The scale factor ρ_a is, formally, the ratio of the asymptotic radius of the free jet boundary to the nozzle exit radius r_j

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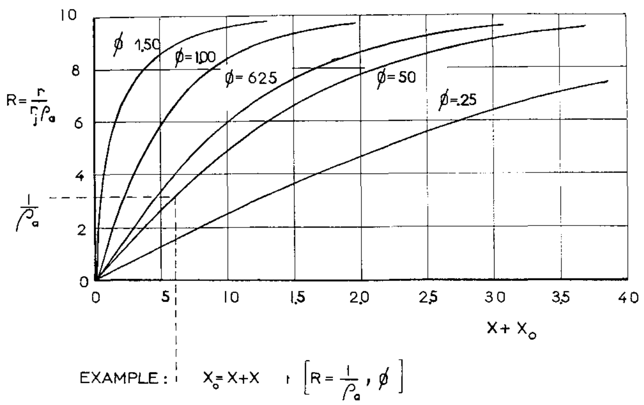

Fig 1 Generalized coordinates of the free jet boundaries

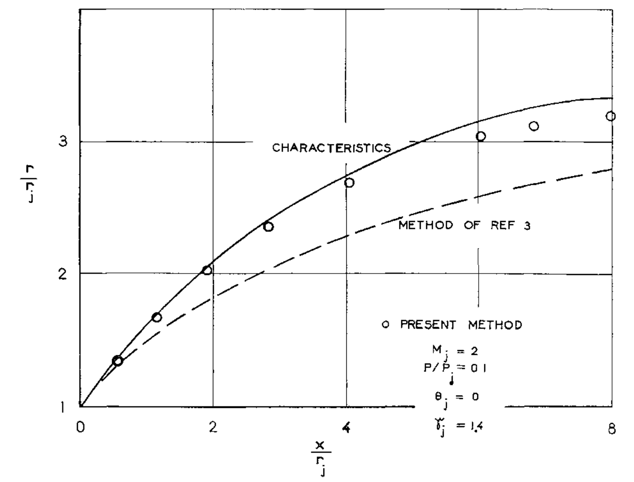

Fig 2 Comparison of characteristics solution[1] with approximate analysis for a typical case

The cellular structure of real jets cannot be reproduced by our model, in which the reflection of characteristics emitted by the opposite edge of the nozzle is absent However, if M_j is not too near unity, one can expect the jet boundary to approach its maximum radius by the turning mechanism only; in that case ρ_a should be nearly equal to the true ratio r_{max}/r_j

Table 1 gives a comparison of the magnitude of ρ_a together with first maximum-radius ratios for several extreme cases reported in Ref 1 The results indicate that a good approximation, particularly at high jet-Mach numbers, is

$$\rho_{max} = r_{max}/r_j \simeq 0\ 96\rho_a \tag{12}$$

for all nozzle divergence angles and underexpansion pressure ratios

The axial location of the first maximum in the jet boundaries should coincide with a value of X at which ρ (or R) reach a given fraction near unity of ρ_a Figure 3 is a crossplot of the present solution on which are shown points corresponding to X_{max} from Ref 1 To a surprising accuracy, throughout the range of jet parameters, the true maximum radius location is given in terms of this model by

$$x_{max}/r_j = \{[(X + X_0) \text{ at } R = 0\ 94] - X_0\}\rho_a \tag{13}$$

The present approximation seems to be consistently better than that of Ref 3 With regard to the empirical correlations suggested in Ref 2, the following can be implied:

1) For $P/P_j = 1$, the present analysis yields a simple closed form solution,

$$\rho_a = \left[1 + \frac{\gamma_j + 1}{2}\theta_j \frac{M_j^2}{(M_j^2 - 1)^{1/2}}\right]^{1/2} \tag{14}$$

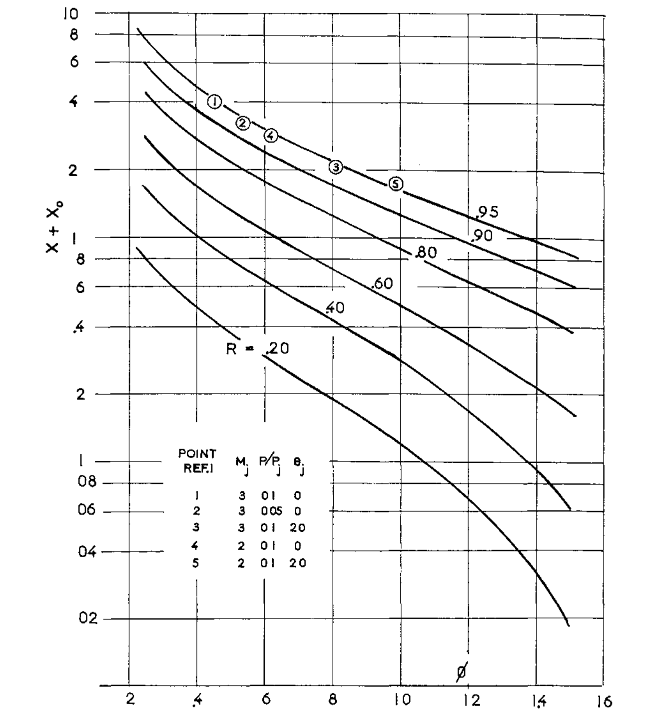

Fig 3 Generalized jet-boundary coordinates Points indicate the axial location of first-maximum jet ratios from characteristics solution [1]

Table 1 Comparison of results for the maximum radius with calculation by the method of characteristics

P/P_j	M_j	θ_j	γ_j	Present analysis, ρ_a	Ref 1, r_{max}/r_j	Ratio,[1] ρ_a/ρ_{max}
0 1	2	0	1 4	3 37	3 35	1
	3			3 50	3 53	0 99
0 1	3	0	1 4	3 50	3 53	0 99
0 05				5 09	4 82	1 05
0 1	3	10	1 4	4 10	3 78	1 085
		20		4 63	4 33	1 07
	2	20		4 23	3 96	1 07
1 0	2	5	1 4	1 115	1 06	1 050
		10		1 216	1 14	1 066
		15		1 311	1 20	1 092
		20		1 402	1 29	1 083

which is to be compared with the correlation suggested in Ref 2, $\rho_{max} \sim 1 + \frac{7}{20}M_j \tan\theta_j$ Although both agree for low M_j, they differ substantially when M_j is large

2) For expansion into near vacuum $P/P_j \to 0$, Ref 2 suggests the variation $\rho_{max} \sim (P/P_j)^{1/2}$ The present model indicates instead,

$$\rho_a \sim \psi^{-1/2} \sim (P/P_j)^{(3 - \gamma_j)/4\gamma_j} \tag{15}$$

The exponent is 0 286 for $\gamma_j = 1\ 4$

There is no evidence that the success of the present solution should be limited to the range over which comparison with calculations of Ref 1 could be made On the contrary, the analysis appears to give a more accurate and better founded framework for correlating and extending data than the purely empirical relations proposed in Ref 2

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Variable-Mesh Difference Equation for the Stream Function in Axially Symmetric Flow

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A finite difference equation is developed for the stream function in cylindrical coordinates with axial symmetry which is applicable to an irregular mesh having different length and radial dimensions In addition, the length and radial dimensions may be varied, and the mesh made finer in any interior region The equation also takes into account an irregular boundary

THE stream function in cylindrical coordinates for the case of axial symmetry is

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{1}$$

A five-point mesh crossing an irregular boundary is used and is shown in Fig 1 The mesh under consideration has spacing of h units in the z direction and k units in the r direction, and α and β are the ratios of the distance to the boundary divided by the mesh distance If the function $\psi(z,r)$ is expanded in a Taylor's series in the r direction, dropping the argument for the derivatives, the following equations result:

$$\psi(z,r+k) = \psi(z,r) + k\frac{\partial\psi}{\partial r} + \frac{k^2}{2!}\frac{\partial^2\psi}{\partial r^2} + \frac{k^3}{3!}\frac{\partial^3\psi}{\partial r^3} + \frac{k^4}{4!}\frac{\partial^4\psi}{\partial r^4} \tag{2}$$

$$\psi(z,r-\alpha k) = \psi(z,r) - \alpha k\frac{\partial\psi}{\partial r} + \frac{\alpha^2k^2}{2!}\frac{\partial^2\psi}{\partial r^2} - \frac{\alpha^3k^3}{3!}\frac{\partial^3\psi}{\partial r^3} + \frac{\partial^4k^4}{4!}\frac{\partial^4\psi}{\partial r^4} + \tag{3}$$

If Eq (2) is multiplied by α and the result added to Eq (3),

$$\frac{\partial^2\psi}{\partial r^2} = \frac{2\psi(z,r+k)}{k^2(1+\alpha)} + \frac{2\psi(z,r-\alpha k)}{k^2\alpha(1+\alpha)} - \frac{2\psi(z,r)}{k^2\alpha^2} + (\alpha^2-1)0k + 0k^2 \tag{4}$$

where $0k$ and $0k^2$ are terms of the order of k and k^2, respectively

If Eq (2) is multiplied by α^2 and the result subtracted from Eq (3), the following is obtained when dividing by r:

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\alpha\psi(z,r+k)}{rk(1+\alpha)} - \frac{\psi(z,r-\alpha k)}{rk\alpha(1+\alpha)} + \frac{(1-\alpha)\psi(z,r)}{rk\alpha} + 0k^2 \tag{5}$$

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In a similar way,

$$\frac{\partial^2\psi}{\partial z^2} = \frac{2\psi(z-h,r)}{h^2(1+\beta)} + \frac{2\psi(z+\beta h,r)}{h^2\beta(1+\beta)} - \frac{2\psi(z,r)}{h^2\beta^2} + (\beta^2-1)0h + 0h^2 \tag{6}$$

Equations (4–6) can be substituted in Eq (1), and the result is a difference form of the stream function for the point $\psi(z,r)$ in terms of the four surrounding points:

$$\psi(z,r+k)\left(\frac{2}{1+\alpha}\right)\left(1-\frac{\alpha k}{2r}\right) + \psi(z,r-\alpha k)\left(\frac{2}{1+\alpha}\right)\left(\frac{1}{\alpha}\right)\left(1+\frac{k}{2r}\right) + \psi(z+\beta h,r)\left[\frac{2\lambda^2}{\beta(1+\beta)}\right] + \psi(z-h,r)\left(\frac{2\lambda^2}{1+\beta}\right) - 2\psi(z,r)\left[\frac{\lambda^2}{\beta} + \frac{1}{\alpha} - \frac{k(1-\alpha)}{2\alpha r}\right] + (1-\beta^2)0h - (1-\alpha^2)0k + 0h^2 + 0k^2 = 0 \tag{7}$$

$$\lambda^2 = k^2/h^2,\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1$$

Equation (7) is valid for any mesh point near a boundary It also applies to an interior point where a change in mesh size is introduced This feature is particularly valuable when evaluating the stream function near an abruptly changing boundary For example, evaluating Eq (7) near a change

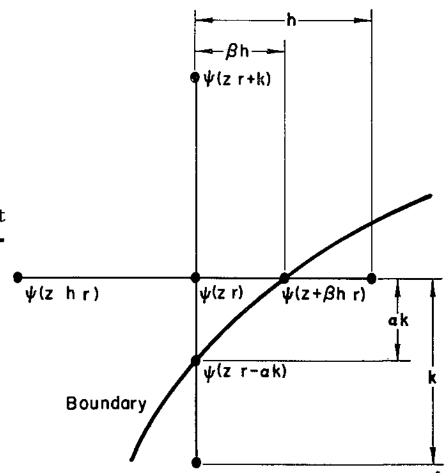

Fig 1 Five-point mesh used in difference equation

in mesh size in the z direction corresponds to a vertical boundary in Fig 1 through the point $\psi(z+h,r)$, and β becomes the ratio of the mesh sizes The mesh need not be square nor regular, that is, k and h need not be equal nor do they always have to be constant

For an interior point in a mesh where $h = k =$ const, Eq. (7) reduces to the familiar form (e g , see Salvadori and Baron[1])

$$\psi(z,r+k)\left(1-\frac{k}{2r}\right) + \psi(z,r-k)\left(1+\frac{k}{2r}\right) + \psi(z-k,r) + \psi(z+k,r) - 4\psi(z,r) = 0 \tag{8}$$

The error involved is of the order of $(1-\beta^2)h$ in Eq (7) Care must be exercised to ensure that β does not approach zero In constructing the net it is necessary to make $(1-\beta^2) \rightarrow h$ if the whole term is to be of the order of h^2 The same argument holds for α If the net is made fine enough, $h, k \ll 1$, and the characteristic dimension of the body under consideration is unity, the terms $0h^2$ and $0k^2$ tend to zero

Also, it is assumed when using Taylor's expansion that all derivatives are bounded This is, of course, not true at the stagnation point of a body of revolution for example However, if the value of the mesh point at the stagnation point